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ANY TWO IRREDUCIBLE MARKOV CHAINS OF EQUAL ENTROPY ARE FINITARILY KAKUTANI EQUIVALENT

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ABSTRACT

We show here that any two finite state irreducible Markov chains of the same entropy are finitarily Kakutani equivalent. By this we mean they are orbit equivalent by an invertible measure preserving mapping that is almost continuous and monotone in time when restricted to some cylinder set. Smorodinsky and Keane have shown that any two irreducible Markov chains of equal entropy and period are finitarily isomorphic. Hence, all that is necessary to obtain our result is to show that for every entropy $h > 0$ and period $p \in \mathbb{N}$ there exists two irreducible Markov chains σ_1, σ_2 both of entropy h, where:

(1) σ_1 is mixing,
(2) σ_2 has period p and
(3) σ_1 and σ_2 are finitarily Kakutani equivalent.

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1. Introduction

A map between two spaces is called **finitary** or **almost continuous** if it is both measure preserving and almost continuous. This concept only makes sense when the underlying spaces are both topological and measure spaces. Thus, finitary theory in dynamics is most natural in classes of dynamical systems that are simultaneously measure preserving and topological.

In 1979 Keane and Smorodinsky [KS1] showed that two irreducible Markov chains of equal entropy and equal period are finitarily isomorphic. This expanded on their earlier work showing all Bernoulli shifts of equal entropy were finitarily isomorphic. More recently, Keane, Hamachi and Roychowdhury have begun the development of a finitary orbit equivalence theory, see [HK], [HKR], [R1] and [R2]. Our work here is part of a larger project to develop a finitary Kakutani equivalence theory. We take the perspective of even Kakutani equivalence that allows us to regard a Kakutani equivalence as an orbit equivalence with the restriction that it be an isomorphism when restricted to some subset of positive measure. Group rotations are natural examples that are both measure preserving and topological and we have begun their study. Irreducible Markov chains are a natural class of examples and now in positive entropy and we completely classify their finitary Kakutani equivalence theory here.

Finitary theory, as indicated above, makes most sense when considered on specific classes of examples and has always been a constructive theory. Our work here is precisely in that tradition. We show that any two finite state irreducible Markov chains of equal entropy are finitarily Kakutani equivalent. The earlier work of Smorodinsky and Keane, showing that entropy and period give complete invariants for finitary isomorphism of irreducible Markov chains, makes our task even more constructive in that all we need to do is to exhibit, for each entropy $h > 0$ and period $p \in \mathbb{N}$ that there exist Markov chains σ_1 and σ_2, both of entropy h, where:

(1) σ_1 is mixing,
(2) σ_2 has period p and
(3) σ_1 and σ_2 are finitarily Kakutani equivalent.

To explain this in detail, suppose T_p and T_q are two irreducible Markov chains of equal entropy h, T_p has period p and T_q has period q. Then, using Keane and Smorodinsky's result we can say that T_p and σ_2 are finitarily isomorphic, and σ_2 is finitarily evenly Kakutani equivalent to a mixing Markov chain σ_1.

Using the same argument there exists an irreducible Markov chain σ_2' of entropy h and period q such that T_q is finitarily isomorphic to σ_2' and σ_2' is finitarily evenly Kakutani equivalent to a mixing Markov chain σ_1'.

Using Keane and Smorodinsky's result again we have that σ_1 and σ_1' are finitarily isomorphic. Since finitarily evenly Kakutani equivalence relation is an equivalence relation, T_p and T_q are finitarily evenly Kakutani equivalent.

2. Construction of Markov chains σ_1 and σ_2

Case 1: σ_1 is mixing and σ_2 has period ≥ 3.

Let

$$\Sigma = \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k, 1, 2, 3, \ldots, p, s_1, s_k\}$$

be the state space of a Markov chain with $(k + p + 2)$ states, $p \geq 2$. Let us denote this Markov chain by the shift automorphism σ. Let $A(i, j)$ represent the transition probability of a state i to a state j. To define the transition matrix A let $(p_1, p_2, \ldots, p_k)$ be a probability vector with all $p_i > 0$ and set:

$$\begin{aligned}
A(1, 2) = A(2, 3) = A(3, 4) = \ldots = A(p - 1, p) = 1 \\
A(p, s_1) = p_1, \quad A(p, \bar{s}_2) = p_2, \quad A(p, \bar{s}_3) = p_3, \ldots, \\
A(p, \bar{s}_{k-1}) = p_{k-1}, \quad A(p, s_k) = p_k \\
A(s_1, \bar{s}_1) = 1 \\
A(s_k, \bar{s}_k) = 1 \\
A(\bar{s}_i, 1) = 1 \quad \forall\, i = 1, 2, \ldots, k - 1 \\
A(\bar{s}_k, 2) = 1
\end{aligned}$$

and this makes σ irreducible and mixing.

A glance at the graph of the chain (shown in Figure 1) shows all states communicate and hence it is irreducible. To see it is mixing note it has two paths

$$p \to s_1 \to \bar{s}_1 \to 1 \to 2 \to \cdots \to p \quad \text{and} \quad p \to s_k \to \bar{s}_k \to 2 \to \cdots \to p,$$

whose lengths differ by 1.

By setting $p_1 = p_k$ a relatively simple symmetry argument gives us that in the Markov chain σ we will have that $\rho(\bar{s}_1) = \rho(\bar{s}_k)$ where $\rho(i)$ is the stationary probability of state i. The structure of the graph of the Markov chain also gives

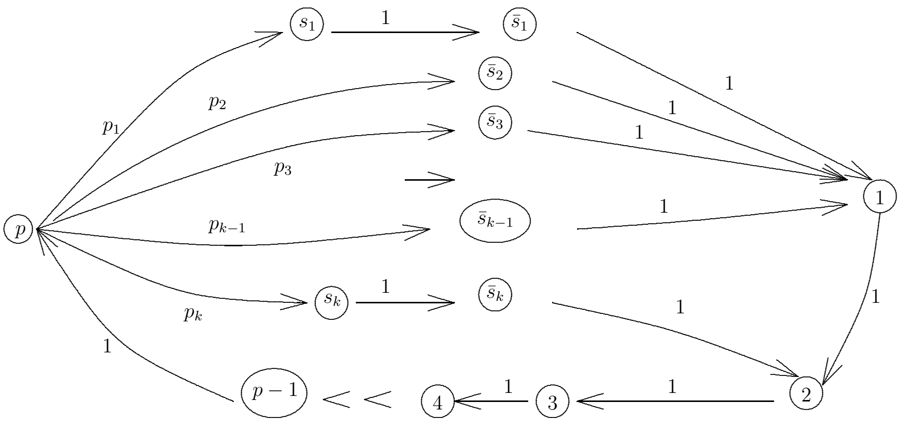

Figure 1

us that

$$\rho(2) = \rho(3) = \cdots = \rho(p),$$
$$\rho(s_1) = \rho(\bar{s}_1),\ \rho(s_k) = \rho(\bar{s}_k),$$
$$\rho(1) = \rho(\bar{s}_1) + \rho(\bar{s}_2) + \cdots + \rho(\bar{s}_{k-1})$$
$$\rho(2) = \rho(\bar{s}_1) + \rho(\bar{s}_2) + \cdots + \rho(\bar{s}_{k-1}) + \rho(\bar{s}_k)$$
$$\rho(s_1) = p_1\rho(p) = \rho(\bar{s}_k), \quad \rho(\bar{s}_2) = p_2\rho(p), \quad \rho(\bar{s}_3) = p_3\rho(p)$$
$$\vdots$$
$$\rho(s_k) = p_k\rho(p).$$

Using the above relations we have,

$$\begin{aligned}
\rho(1) + \rho(2) + \cdots + \rho(p) + \rho(s_1) + \rho(\bar{s}_1) + \rho(\bar{s}_2) + \cdots + \rho(\bar{s}_k) + \rho(s_k) = 1 \\
\implies (p-1)\rho(p) + \rho(1) + \rho(s_1) + \rho(2) + \rho(s_k) = 1 \\
\implies (p-1)\rho(p) + (\rho(1) + \rho(s_k)) + \rho(s_1) + \rho(2) = 1 \\
\implies (p-1)\rho(p) + \rho(2) + \rho(s_1) + \rho(2) = 1 \\
[\because \rho(2) - \rho(1) = \rho(\bar{s}_k) = \rho(s_k)] \\
\implies (p-1)\rho(p) + \rho(p) + p_1\rho(p) + \rho(p) = 1 \\
\implies \rho(p) = \frac{1}{p+1+p_1}.
\end{aligned}$$

Hence, the entropy of σ is given by

$$\begin{aligned}
h(\sigma) =& - \sum_{\text{pairs of } s_1, s_2} \rho(s_1) A(s_1, s_2) \log_2(A(s_1, s_2)) \\
=& - [\rho(p)A(p, s_1)\log_2 A(p, s_1) + \rho(p)A(p, \bar{s}_2)\log_2 A(p, \bar{s}_2) + \cdots \\
& + \rho(p)A(p, \bar{s}_{k-1})\log_2 A(p, \bar{s}_{k-1}) + \rho(p)A(p, s_k)\log_2 A(p, s_k)] \\
=& - \rho(p)\Sigma_{i=1}^{k} p_i \log_2 p_i \\
=& - \frac{1}{p+1+p_1}\Sigma_{i=1}^{k} p_i \log_2 p_i.
\end{aligned}$$

Let $\sigma_1 = \sigma_{\bar{s}_k^c}$ be the induced transformation of σ on the cylinder set $\bar{s}_k^c$ i.e. after removing $\bar{s}_k$ from σ. As the state $\bar{s}_k$ always follows the state s_k the induced transformation is generated by the remaining symbols and in these terms σ_1 is still irreducible. In addition, it is mixing since it has two paths

$$p \to \bar{s}_{k-1} \to 1 \to 2 \to \cdots \to p \quad \text{and} \quad p \to s_k \to 2 \to \cdots \to p$$

whose lengths differ by 1.

Let $\sigma_2 = \sigma_{\bar{s}_1^c}$ be the induced transformation of σ on $\bar{s}_1^c$ i.e. after removing $\bar{s}_1$ from σ. As the state $\bar{s}_1$ always follows s_1 the dynamical system after inducing on this complement is generated by the remaining states of the Markov chain and in these terms σ_2 is irreducible and has period $p+1$.

We can calculate the entropy of σ_1 to be

$$\begin{aligned}
h(\sigma_1) =& \frac{1}{1-\rho(s_1)} \frac{1}{p+1+p_1}\left(-\sum p_i \log_2 p_i\right) \\
=& \frac{\rho(p)}{1-p_1\rho(p)}\left(-\sum p_i \log_2 p_i\right) \\
=& \frac{1}{p+1}\left(-\sum p_i \log_2 p_i\right) \\
=& h(\sigma_2)
\end{aligned}$$

the entropy induced on σ_2, i.e. both σ_1 and σ_2 have the same entropy.

We now describe the other possible cases, which are variations on this same idea.

Case 2: σ_1 is mixing and σ_2 has period 2.

Set the state space to be

$$\Sigma = \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k, 1, s_1\},$$

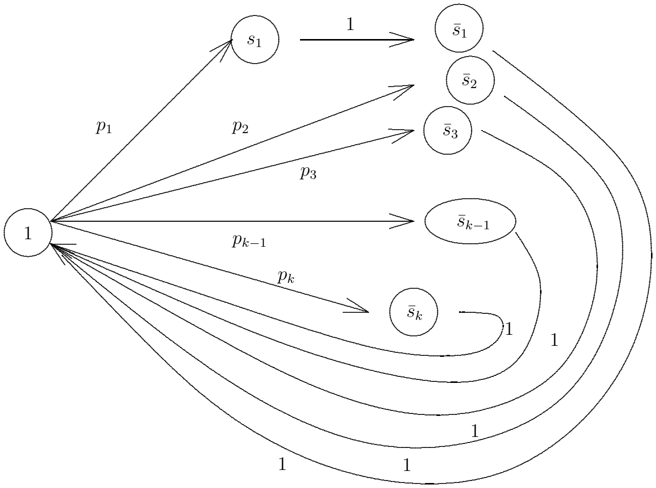

Figure 2

with the transition probabilities:

$$A(1, s_1) = p_1, A(s_1, \bar{s}_1) = A(\bar{s}_i, 1) = 1 \text{ for } i = 1, 2, \ldots, k$$
$$A(1, \bar{s}_i) = p_i \text{ for } i = 2, 3, \ldots, k.$$

One checks easily that the stationary probabilities for this Markov chain are

$$\rho(s_1) = \rho(\bar{s}_1)$$
$$\rho(s_1) = p_1\rho(1)$$
$$\rho(\bar{s}_2) = p_2\rho(1)$$
$$\vdots$$
$$\rho(\bar{s}_k) = p_k\rho(1).$$

We also assume $\rho(\bar{s}_1) = \rho(\bar{s}_k)$.

Using the above relations we have,

$$\rho(1) + \rho(s_1) + \rho(\bar{s}_1) + \rho(\bar{s}_2) + \cdots + \rho(\bar{s}_k) = 1$$
$$\implies \rho(1) + \rho(1) + \rho(\bar{s}_1) = 1$$
$$\implies 2\rho(1) + p_1\rho(1) = 1$$
$$\implies \rho(1) = 1/p_1 + 2.$$

Denote the Markov chain with Markov measure μ and the shift automorphism σ. Then proceeding as Case 1, it is easy to show that induced Markov chain $\sigma_1 = \sigma_{\bar{s}_k^c}$ is irreducible and mixing, the induced Markov chain $\sigma_2 = \sigma_{\bar{s}_1^c}$ is irreducible and has period 2 and both σ_1 and σ_2 have the same entropy.

3. Constructing the finitary Kakutani equivalence

We have constructed an initial Markov chain σ and then induced on the complements of two subsets $\bar{s}_1$ and $\bar{s}_k$ to build the two Markov chains σ_1 and σ_2. Now both $\bar{s}_1$ and $\bar{s}_k$ have the same measure. Our goal is to show σ_1 and σ_2 are almost continuously Kakutani equivalent. We prove this by working on the global space X on which σ acts. What we will do is to construct a finitary isomorphism $\phi : X \to X$ that carries $\bar{s}_1$ to $\bar{s}_2$, acts as the identity off of these two cylinder sets and preserves orbits. It will be convenient for us to make ϕ an involution as then there is no question of it being invertible and possessing a finitary inverse. We now begin this construction.

We let Σ^n represent all words of length n allowed by the Markov matrix A and $\bigcup_{n=1}^{\infty} \Sigma^n$ represents the set of all possible words in the Markov chain. For any word $w = [\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_p] \in \bigcup_{n=1}^{\infty} \Sigma^n$ by $B(w)$ we denote

$$B(w) = \{\vec{x} : x_0 = \bar{w}_0, x_1 = \bar{w}_1, \ldots, x_p = \bar{w}_p\}.$$

Let us choose the word $w_0 = [p, \bar{s}_2, 1, 2, \ldots, p-1]$ and by B_k we denote the $k+1$-fold concatenation of copies of w_0 i.e. $B_k = B(w_0^{k+1})$ $(k \geq 0)$.

For a.e. $x \in X$, $\exists\ i_0(x) \leq 0$, $j_0(x) > 0$ s.t.

$$i_0(x) = \max\{t \leq 0 : \sigma^t(x) \in B_0\}, \quad j_0(x) = \min\{t > 0 : \sigma^t(x) \in B_0\}.$$

Then set $[B_0, B_0)_x = \{\sigma^{i_0(x)}(x), \sigma^{i_0(x)+1}(x), \ldots, x, \sigma(x), \ldots, \sigma^{j_0(x)-1}(x)\}$ arranged in increasing order as listed here according to the power of σ as

$$\sigma^{i_0(x)}(x) < \sigma^{i_0(x)+1}(x) < \cdots < x < \sigma(x) < \cdots < \sigma^{j_0(x)-1}(x).$$

If $\sigma^k(x) \in [B_0, B_0)_x$, then $[B_0, B_0)_{\sigma^k(x)} = [B_0, B_0)_x$ since $i_0(\sigma^k(x)) = i_0(x) - k$ and $j_0(\sigma^k(x)) = j(x) - k$.

We are now ready to begin to define ϕ. We will work inductively defining ϕ on an increasing sequence of sets and here is the 0-th step. First define ϕ_0 by $\phi_0(x) = x$ if $x_0 \neq \bar{s}_1, \bar{s}_k$. This defines ϕ on the clopen set given by the complement of the two sets $\bar{s}_1$ and $\bar{s}_k$ as the identity.

Let

$$N_0(x) = \{t : i_0(x) \le t < j_0(x), (\sigma^t(x))_0 = \bar{s}_1\},$$
$$M_0(x) = \{t : i_0(x) \le t < j_0(x), (\sigma^t(x))_0 = \bar{s}_k\}.$$

Set $P_0(x) = \min\{\#N_0(x), \#M_0(x)\}$. List $N_0(x)$ and $M_0(x)$ in order as

$$N_0(x) = \{t_1 < t_2 < \cdots < t_l < \cdots < t_{\#N_0(x)}\}$$

and

$$M_0(x) = \{s_1 < s_2 < \cdots < s_l < \cdots < s_{\#M_0(x)}\}.$$

Now for almost every $x \in X$ we have $x \in [B_0, B_0)_x$ and if $x_0 = \bar{s}_1$, then $0 = t_l$ for some $t_l \in N_0(x)$.

If $l \le P_0(x)$, define $\phi_0(x) = \sigma^{s_l}(x)$ and $\phi_0(\sigma^{s_l}(x)) = x$. This extends the definition of ϕ_0 to some points whose zeroth coordinate is equal to $\bar{s}_1$ or $\bar{s}_k$.

We examine this definition a bit. Fixing a set $[B_0, B_0)_x$ and considering the various points $\sigma^j(x)$ in it we see $\phi_0(\sigma^{t_i}(x)) = \sigma^{s_i}(x)$ and $\phi_0(\sigma^{s_i}(x)) = \sigma^{t_i}(x)$ for $i = 1, 2, \ldots, P_0(x)$.
Thus where it is defined ϕ_0 preserves σ orbits, is the identity off $\bar{s}_1 \cup \bar{s}_k$, carries $\bar{s}_1$ to $\bar{s}_k$ and acts as an involution where it is defined.

Continuing inductively we define:

$$i_1(x) = \max\{t \le 0 : \sigma^t(x) \in B_1\}, \quad j_1(x) = \min\{t > 0 : \sigma^t(x) \in B_1\}.$$

Then $[B_1, B_1)_x = \{\sigma^{i_1(x)}(x), \sigma^{i_1(x)+1}(x), \ldots, x, \sigma(x), \ldots, \sigma^{j_1(x)-1}(x)\}$ arranged in increasing order according to the power of σ as

$$\sigma^{i_1(x)}(x) < \sigma^{i_1(x)+1}(x) < \cdots < x < \sigma(x) < \cdots < \sigma^{j_1(x)-1}(x).$$

Let

$$N_1(x) = \{t : i_1(x) \le t < j_1(x),\ (\sigma^t(x))_0 = \bar{s}_1 \text{ and } \phi_0 \text{ is not defined at } \sigma^t(x)\}$$

and

$$M_1(x) = \{t : i_1(x) \le t < j_1(x),\ (\sigma^t(x))_0 = \bar{s}_k \text{ and } \phi_0 \text{ is not defined at } \sigma^t(x)\}.$$

Now for almost every x we have $x \in [B_1, B_1)_x$, and we define ϕ_1 as follows:

(1) If ϕ_0 is defined at x, then $\phi_1(x) = \phi_0(x)$.

(2) If ϕ_0 is not defined at x, then we must have $x_0 = \bar{s}_1$ or $x_0 = \bar{s}_k$, and we attempt to define ϕ_1 as follows: List $\{\sigma^t(x) : t \in N_1(x)\}$ in order $x_1 < x_2 < x_3 < \cdots < x_{\#N_1(x)}$. List $\{\sigma^t(x) : t \in M_1(x)\}$ in order $y_1 < y_2 < y_3 < \cdots < y_{\#M_1(x)}$. Set $P_1(x) = \min\{\#N_1(x), \#M_1(x)\}$. Now for $1 \le i \le P_1(x)$ define $\phi_1(x_i) = y_i,\ \phi_1(y_i) = x_i$.

Proceeding inductively for any positive integer k define:

$$i_k(x) = \max\{t \le 0 : \sigma^t(x) \in B_k\}$$

and

$$j_k(x) = \min\{t > 0 : \sigma^t(x) \in B_k\}.$$

Now set $[B_k, B_k)_x = \{\sigma^{i_k(x)}(x), \sigma^{i_k(x)+1}(x), \ldots, x, \sigma(x), \ldots, \sigma^{j_k(x)-1}(x)\}$ arranged in order according to the power of σ as

$$\sigma^{i_k(x)}(x) < \sigma^{i_k(x)+1}(x) < \cdots < x < \sigma(x) < \cdots < \sigma^{j_k(x)-1}(x).$$

Let:

$$N_k(x) = \\ \{t : i_k(x) \le t < j_k(x),\ (\sigma^t(x))_0 = \bar{s}_1 \text{ and } \phi_{k-1} \text{ is not defined at } \sigma^t(x)\}$$

and

$$M_k(x) = \\ \{t : i_k(x) \le t < j_k(x),\ (\sigma^t(x))_0 = \bar{s}_k \text{ and } \phi_{k-1} \text{ is not defined at } \sigma^t(x)\}.$$

Now for almost every x we have $x \in [B_k, B_k)_x$, we attempt to define $\phi_k(x)$ as:

(1) If ϕ_{k-1} is defined at x, then $\phi_k(x) = \phi_{k-1}(x)$.
(2) If ϕ_{k-1} is not defined at x, then we must have $x_0 = \bar{s}_1$ or $x_0 = \bar{s}_k$, and we attempt to define $\phi_k(x)$ in the following way. List $\{\sigma^t(x) : t \in N_k(x)\}$ in order $x_1 < x_2 < x_3 < \cdots < x_{\#N_k(x)}$. List $\{\sigma^t(x) : t \in M_k(x)\}$ in order $y_1 < y_2 < y_3 < \cdots < y_{\#M_k(x)}$. Set $P_k(x) = \min\{\#N_k(x), \#M_k(x)\}$. Now for $1 \le i \le P_k(x)$ define $\phi_k(x_i) = y_i,\ \phi_k(y_i) = x_i$.

This now completes the definition of ϕ as

$$\phi = \bigcup \phi_i \quad \text{s.t.} \quad \phi(x) = \phi_i(x), \quad \text{if} \quad \phi_i \text{ is defined at } x.$$

We now complete our understanding of ϕ through a series of lemmas.

Lemma 4: *ϕ is defined at a.e. $x \in X$.*

Proof. Let $E = \{x : \phi \text{ is not defined at } x\}$. We will show that $\mu(E) = 0$.

Suppose $\mu(E) = \alpha > 0$. Then $\forall\ k$,

$$\mu(\{x : \phi_k \text{ is not defined at } x\}) \geq \alpha > 0.$$

We know that for a.e. x, $i_k(x)$ and $j_k(x)$ are finite and

$$\lim_{k\to\infty} i_k(x) = -\infty, \ \lim_{k\to\infty} j_k(x) = \infty.$$

Again, by Birkhoff's ergodic theorem,

$$\lim_{k\to\infty} \frac{1}{|i_k(x)|+1} \sum_{t=0}^{|i_k(x)|} \chi_E(\sigma^{-t}(x)) = \mu(E) = \alpha$$
$$\lim_{k\to\infty} \frac{1}{j_k(x)} \sum_{t=1}^{j_k(x)} \chi_E(\sigma^{t}(x)) = \mu(E) = \alpha.$$

Now

$$\begin{aligned}\sum_{t=-|i_k(x)|}^{j_k(x)} \chi_E(\sigma^t(x)) &= \sum_{t=-|i_k(x)|}^{0} \chi_E(\sigma^t(x)) + \sum_{t=1}^{j_k(x)} \chi_E(\sigma^t(x))\\ &= \sum_{t=0}^{|i_k(x)|} \chi_E(\sigma^{-t}(x)) + \sum_{t=1}^{j_k(x)} \chi_E(\sigma^t(x))\\ &= (\alpha \pm \epsilon_k)(|i_k(x)|+1) + (\alpha \pm \epsilon_k)(j_k(x))\\ &= (\alpha \pm \epsilon_k)(|i_k(x)| + j_k(x) + 1).\end{aligned}$$

This implies for a.e. x,

$$\lim_{k\to\infty} \frac{1}{|i_k(x)| + j_k(x) + 1} \sum_{t=-|i_k(x)|}^{j_k(x)} \chi_E(\sigma^t(x)) = \alpha.$$

Now for every value of k,

$$\begin{aligned}\#\{t : -|i_k(x)| \le t < j_k(x), \phi_k(x) \text{ in not defined at } \sigma^t(x)\}\\ =&|\#\{t : -|i_k(x)| \le t < j_k(x),\ (\sigma^t(x))_0 = \bar{s}_1\}\\ &- \#\{t : -|i_k(x)| \le t < j_k(x),\ (\sigma^t(x))_0 = \bar{s}_k\}|\\ =&\left|\sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_1}(\sigma^t(x)) - \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_k}(\sigma^t(x))\right|\\ =&(|i_k(x)| + j_k(x))\left|\frac{1}{|i_k(x)| + j_k(x)} \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_1}(\sigma^t(x))\right.\\ &\left. - \frac{1}{|i_k(x)| + j_k(x)} \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_k}(\sigma^t(x))\right|.\end{aligned}$$

Therefore, for every value of k

$$\begin{aligned}&\frac{1}{|i_k(x)| + j_k(x)} \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_E(\sigma^t(x))\\ &= \frac{1}{|i_k(x)| + j_k(x)}\#\{t : -|i_k(x)| \le t < j_k(x),\ \phi \ \text{ is not defined at } \sigma^t(x)\}\\ &\le \frac{1}{|i_k(x)| + j_k(x)}\#\{t : -|i_k(x)| \le t < j_k(x),\ \phi_k \ \text{ is not defined at } \sigma^t(x)\}\\ &= \left|\frac{1}{|i_k(x)| + j_k(x)} \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_1}(\sigma^t(x)) - \frac{1}{|i_k(x)| + j_k(x)} \sum_{t=-|i_k(x)|}^{j_k(x)-1} \chi_{\bar{s}_k}(\sigma^t(x))\right|\\ &\to |\mu(\bar{s}_1) - \mu(\bar{s}_k)| = 0.\end{aligned}$$

Which implies $\alpha = 0$, which is a contradiction and so ϕ is defined a.e. ∎

LEMMA 5: *For a.e. $x \in X$, x and $\phi(x)$ are in the same orbit.*

Proof. For any $x \in X$, there exists an integer $k \ge 0$ with $\phi(x) = \phi_k(x)$. In this case both x and $\phi_k(x)$ belong to $[B_k, B_k)_x$ which is a finite subset of the orbit of x. ∎

LEMMA 6: *ϕ is an involution and hence invertible.*

Proof. ϕ is defined at a.e. $x \in X$. If ϕ is defined at x then $\exists\ k \geq 0$ such that $\phi(x) = \phi_k(x)$ and by construction $\phi_k(\phi_k(x)) = x$. Hence for a.e. $x, \phi(\phi(x)) = x$. ∎

LEMMA 7: *ϕ is almost continuous and measure preserving.*

Proof. For any integer $k \geq 0$, we define a relation R_k in X as follows:

For any two elements $x, y \in X$, xR_ky holds if $i_k(x) = i_k(y)$ and $j_k(x) = j_k(y)$ and

$$(x_{i_k(x)}, x_{i_k(x)+1}, \ldots, x_{j_k(x)-1}) = (y_{i_k(y)}, y_{i_k(y)+1}, \ldots, y_{j_k(y)-1}).$$

That is to say, two points x, y are R_k related if they exhibit the same symbolic names across the sequence of points in $[B_k, B_k)_x$ and $[B_k, B_k)_y$, and each sits at the same position within the block. This is clearly an equivalence relation and partitions X into mutually disjoint clopen equivalence classes, each being a cylinder set given by a fixed word across indices $i_k(x)$ through $j_k(x) - 1$. Let $B_1^k, \ldots, B_{n(k)}^k$ be the equivalence classes of R_k. By the construction we see that on each set B_i^k either ϕ_k is not defined on any points of B_i^k or it is some fixed power $\sigma^{n(i,k)}$ on all points in B_i^k. Now, the equivalence classes of R_{k+1} refine those of B_i^k and extend the definition of ϕ_k to some further cylinders. Thus, almost all of X can be partitioned into clopen subsets on each of which ϕ acts as a constant power of σ. It follows that ϕ is almost continuous and, in particular, measurable.

Any measurable and invertible transformation that acts a.e. as a power of σ must be measure preserving. To see this partition X into subsets X_t on which ϕ acts as σ^t. Restricted to each X_t the map ϕ is measure preserving. As the sets $\phi(X_t)$ must also partition X a.s. we see that ϕ itself is measure preserving. ∎

THEOREM 8: *σ_1 and σ_2 are finitarily evenly Kakutani equivalent.*

Proof. Recall that $\phi : \bar{s}_1 \to \bar{s}_k$, $\phi : \bar{s}_k \to \bar{s}_1$ and ϕ is identity on $(\bar{s}_1 \cup \bar{s}_k)^c$ and carries orbits of σ to themselves. Hence, $\phi : \bar{s}_1^c \to \bar{s}_k^c$ and $\phi : \bar{s}_k^c \to \bar{s}_1^c$ which implies $\phi(\sigma_1)\phi^{-1}$ has the same orbits as σ_2 and as ϕ is almost continuous, ϕ is a finitary orbit equivalence between these two transformations.

Now consider the two induced maps $\sigma_{1,\bar{s}_1^c} = \sigma_{(\bar{s}_1 \cup \bar{s}_k)^c}$ and $\sigma_{2,\bar{s}_k^c} = \sigma_{(\bar{s}_1 \cup \bar{s}_k)^c}$. These two transformations are obviously the same but, more importantly, the map ϕ restricts to an isomorphism (the identity) between them. Hence ϕ is an almost continuous Kakutani equivalence between the two Markov chains σ_1 and σ_2. ∎

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